EQUATIONS OF MECHANICS FOR GAS-SATURATED POROUS MEDIA

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A gas-saturated porous body is considered as a microinhomogeneous medium consisting of a homogeneous linearly elasic matrix and ellipsoidal pores filled by a gas at pressure p. The positions of the centers and orientations of the inclusions have a Poisson distribution in each fixed region of the medium. The parameters of the medium are statistically homogeneous and ergodic in a macroregion W with dimensions significantly exceeding the characteristic dimensions of the inhomogeneity. The results of a significant number of studies evaluating the properties of such a medium are either invariant relaitve to the form of the pores [1, 2] or do not consider multiparticle interaction of the inclusions with sufficient accuracy [3]. In the widely used effective field method [4] the interaction of inclusions is considered by summation of fields from each point singularity located within some effective field, the structure of which is independent of the properties of the inclusion considered. The present study will present a generalization of the method in which any finite number of inclusions are located in the effective field, so that upon each inclusion there acts a stress field which is dependent on the properties of the inclusion considered. The binary interaction of the inclusions is constructed by the asymptotically exact method of successive approximations. The effective properties of the gas-saturated medium and the stress concentration near individual inclusions are evaluated.

1. General Relationships. We will consider a macroregion W, consisting of a matrix with modulus of elasticity tensor L and Poisson set $X = (V_k, x_k, a_k, w_k)$ of ellipsoidal pores v_k with characteristic functions V_k, centers x_k, semiaxes $a_k^i (i = 1, 2, 3)$ and set of Euler angles $_k$. The current gas pressure in the pores p and the matrix modulus L are assumed constant within the macroregion W, the dimensions of which are significantly smaller than the characteristic dimensions of the construction or region considered. The relationship between stresses and deformations at a micropoint within the medium can be represented in the form

$$\sigma = L_0(1 - V)\varepsilon - qV, \tag{1.1}$$

where $V = \bigcup_k V_k$; $q = p\delta_{ij}$. Substituting Eq. (1.1) in the equilibrium equation, we obtain

$$\nabla L_0 \nabla u = (\nabla L_0 \nabla u + \nabla q)V. \tag{1.2}$$

Here u(x) is the displacement; ∇ is the symmetric gradient operator. Let a homogeneous stress field σ^0 be specified at infinity, whereupon Eq. (1.2) may be reduced to an integral equation

$$u = u^0 - \int U(x - y)(\nabla L_0 \nabla u + \nabla q) V(y)\, dy \tag{1.3}$$

(here U is the Green's tensor of the Lamé equation of a homogeneous medium with elasticity tensor L_0 and dislacement u^0 at infinity). After application of the operator ∇ to Eq. (1.3) and transformation of the integral by Green's theorem we center the equation obtained, i.e., subtract from both sides their averages over the ensemble X;

$$\varepsilon(x) = \langle\varepsilon\rangle - \int G(x - y)\{[L_0\varepsilon(y) + q]V(y) - [L_0\langle\varepsilon V\rangle + q\langle V\rangle]\}\, dy, \tag{1.4}$$

where it has been considered that at sufficient removal x from the boundary ∂W the surface integral operation can be regarded as averaging; here and below $\langle\cdot\rangle$, $\langle\cdot|x_2; x_1\rangle$ denote the average and conditional average over the set X, where at the points x_1, x_2 there are located inclusions $x_1 \neq x_2$, $G = \nabla\nabla U$. As $|x - y| \to \infty$ in Eq. (1.4) the expression in curly brackets vanishes and the integral converges absolutely over the entire integration region.

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 106-109, July-August, 1986. Original article submitted July 17, 1985.

To determine the effective elasticity tensor L^* and the "gas" expansion coefficient β^* in the equation of the macrostate

$$\langle\sigma\rangle = L^*(\langle\varepsilon\rangle - \beta^* q) \quad (1.5)$$

it is necessary to evaluate the tensors B, β^*:

$$\langle\varepsilon V\rangle = B\langle\varepsilon\rangle, \ \langle\varepsilon\rangle = \beta^* q \quad (1.6)$$

at $p = 0$, $\sigma^0 \equiv \langle\sigma\rangle = L_0\nabla u^0 \neq 0$ and $p \neq 0$, $\sigma^0 = 0$, so that

$$L^* = L_0(I - B). \quad (1.7)$$

In Eq. (1.5) the value of q is proportional to the current value of the gas pressure in the pores, which is related in an obvious manner to the specified and easily experimentally determined mean volume gas concentration c in the macroregion W by the Henry and Mandeleev–Clapeyron laws

$$p = c[(1 - \langle V\rangle)(1 + \langle\varepsilon_{ii}\rangle - \langle\varepsilon_{ii}V\rangle)\Gamma + \langle V\rangle(1 + \langle\varepsilon_{ii}V\rangle)\mu/RT]^{-1}, \quad (1.8)$$

where the first term with Henry constant Γ describes the contribution of the mean concentration of gas dissolved in the solid phase, and the second considers the presence in the pore phase of gas with a molecular weight μ at temperature T; R is the universal gas constant. Equation (1.8) can be generalized to a gas mixture in an obvious manner.

Thus to obtain Eq. (1.5) it is necessary to evaluate the mean deformation of the pore phase $\langle\varepsilon V\rangle$ under the action of the applied external stress σ^0 and the gas pressure, which depends on $\langle\varepsilon_{ii}V\rangle$.

2. Effective Field. We will fix an arbitrary realization of X and consider the effective field $\bar{\varepsilon}(x)$, $x \in v_k$ in which an inclusion v_k is located:

$$\bar{\varepsilon}_k(x) = \langle\varepsilon\rangle - \int G(x-y)\{[L_0\varepsilon(y) + q]V(y;x) - [L_0\langle\varepsilon V\rangle + q\langle V\rangle]\}\,dy \ (V(y;x) = V(y)\setminus V_k(x)). \quad (2.1)$$

Since the field X is random $\bar{\varepsilon}_k(x)$ is also random. To find the mean over the set X $\langle\bar{\varepsilon}_k\rangle$ we use the hypotheses: 1) the field $\bar{\varepsilon}_k$ is homogeneous in the vicinity of the inclusion v_k and depends on the dimensions and orientation of v_k; 2) each n (n > 1) of the inclusions $v_1, \ldots, v_n$ is located within its own effective field, $\hat{\varepsilon}_{1,\ldots,n}$, which is independent of the properties of the inclusions considered.

The homogeneous field $\bar{\varepsilon}_k(x)$ of Eq. (2.1) uniquely defines the deformations of the k-th inclusion

$$\varepsilon^+ = \bar{A}_k(\bar{\varepsilon}_k + P_k q), \ A_k = (I - P_k L_0)^{-1}, \quad (2.2)$$

where $P_k = -\int G(x-y)V_k(y)dy \ (x \in v_k)$ does not depend on x and the dimensions of v_k [5]. The limiting value of the deformation tensor in the matrix near the boundary of the ellipsoid at the point $x_0 \in \partial v_k$ with external normal unit vector n to ∂v_k is defined by the expression

$$\varepsilon^-(n) = (I - K_k(n)L_0)A_k\bar{\varepsilon}_k + (P_k - K_k(n))A_k q. \quad (2.3)$$

Here $K_k(n)$ is the change in $P_k(x)$ at the point $x_0 \in \partial v_k$ upon transition through ∂v_k in the direction n, known for an isotropic matrix [6]. From Eqs. (1.1), (1.2) with consideration of hypothesis 1 we find

$$\bar{\varepsilon}_k(x) = \langle\varepsilon\rangle - \int G(x-y)\{A(y)[L_0\bar{\varepsilon}(y) + q]V(y;x) - [L_0\langle A\bar{\varepsilon}V\rangle + \langle AV\rangle q]\}\,dy. \quad (2.4)$$

3. Evaluation of Binary Inclusion Interaction. In Eq. (2.4) it is necessary to evaluate $\bar{\varepsilon}(y)$ in the vicinity of the inclusions $v_m \ni y$ given that at point x there is an inclusion v_k. We will assume that in the macroregion W there are only two inclusions:

$$\varepsilon_k(x) = \varepsilon^0 - \int G(x-y)[L_0\varepsilon(y) + q](V_k(y) + V_m(y))\,dy. \qquad (3.1)$$

We solve Eq. (3.1) by the successive-approximation method with consideration of hypothesis 1 and the zeroth approximation $\varepsilon_0(x) = 0$, $x \in v_k$:

$$(L_0\varepsilon(x) + \bar{q})\bar{v}_k = -R_k J_{km}\varepsilon^0 + F_k + R_k T_{km},\ x \in v_k, \qquad (3.2)$$

$$S(x_k - x_m)(L_0\varepsilon(x) + q)\bar{v}_m = -J_{km}\varepsilon^0 + \varepsilon^0 + T_{km},\ x \in v_m,$$

$$S(x_k - x_m) = (\bar{v}_k\bar{v}_m)^{-1}\int\int V_k(x)V_m(y)G(x-y)\,dxdy,$$

$$R_m = -A_m L_0\bar{v}_m,\ F_m = A_m q\bar{v}_m,\ \bar{v}_m = \text{mes}\, v_m;$$

$$J_{km} = \sum_{i=0}^{\infty}\sum_{j=0}^{1}(SR_m SR_k)^i (SR_m)^j, \qquad (3.3)$$

$$T_{km} = \sum_{i=0}^{\infty}\sum_{j=0}^{1}(SR_m SR_k)^i (SR_m)^j S(F_m)^j (F_k)^l,$$

$k = 1, 2$, $m = 3 - k$, $l = |1 - j|$. To proceed further it will be necessary to evaluate $\langle\langle J_{km}\rangle\rangle_{km}$, $\langle\langle T_{km}\rangle\rangle_{km}$, where $\langle\langle\cdot\rangle\rangle_{km}$ is the operation of averaging over orientations ω_k, ω_m and positions x_m on a sphere of radius $|r| = |x_k - x_m|$ with center at x_k.

4. Evaluation of L^*, β^*. We will describe the structure of the composite of the binary distribution function $\phi(v_m|v_k)$, the probability of location of an inclusion v_m in the region v_m for a fixed inclusion v_k. Since inclusions do not intersect each other, we take

$$\varphi(v_m|v_k) = \psi(\omega_m)(1 - V'_{km})f_{km}(|r|)(\text{mes}\,W)^{-1}, \qquad (4.1)$$

where from the normalization condition $\langle\psi(\omega_m)\rangle = 1$, $f_{km}(|r|) = n_\nu$ if $v_m \in X_\nu$ (where n_ν is the numerical concentration of inclusions of the ν-th pore size X_ν); V'_{km} is the characteristic function of a sphere with center at x_k and radius $a_{km} = \min_i a_m^i + \max_i a_k^i$. We average Eq. (2.4) over the set $X(\cdot|v_k)$ with the aid of (4.1):

$$\langle\bar{\varepsilon}_k\rangle = \langle\varepsilon\rangle - \int G(x-y)\,\{\langle A(y)\,[L_0\bar{\varepsilon}(y) + q]\,V(y;\ x)|\ y;\ x\rangle - [\langle R\bar{\varepsilon}\rangle + \langle F\rangle]\}\,dy. \qquad (4.2)$$

To calculate the moments in Eq. (4.2) we use hypotheses 2 with n = 2 and the assumption $\hat{\varepsilon}_{12} = \hat{\varepsilon} = \text{const}$. Averaging Eq. (4.2) over values of ω_k and a_{km} with use of Eq. (3.2) and $\hat{\varepsilon}_k = \hat{\varepsilon}$ we obtain

$$\langle\hat{\varepsilon}\rangle = D\Big(\langle\varepsilon\rangle - \int\langle\langle(T_{km} - SF_m - G(y)F_m V'_{km}(y))f_{km}\rangle\rangle_{km}\,dy\Big), \qquad (4.3)$$

$$D = \Big(I - \int\langle\langle\big(J_{km} - I - SR_m - G(y)R_m V'_{km}(y)\big)f_{km}\rangle\rangle_{km}dy\Big)^{-1}.$$

From Eqs. (2.2) and (4.3) we find the mean deformation of the pore phase

$$\langle\varepsilon V\rangle = D\langle AV\rangle[\langle\varepsilon\rangle + L_0^{-1}q] - L_0^{-1}\langle V\rangle q. \qquad (4.4)$$

Substituting Eq. (4.4) in Eq. (1.6), (1.7), (2.3) we define

$$L^* = L_0(I - D\langle AV\rangle),\ \beta^* = (L^*)^{-1} - L_0^{-1}, \qquad (4.5)$$

$$\langle\varepsilon^-(n)\rangle = \Big\{(I - K_k(n)L_0)A_k\langle\varepsilon\rangle + (P_k - K_k(n))A_k q - \int\langle\langle(T_{km} - SF_m - G(y)F_m V'_{km}(y))f_{km}\rangle\rangle_{km}dy\Big\}D.$$

We define the pressure p for known L^*, β^* by simultaneous solution of Eqs. (1.5), (1.8), (4.4), (4.5).

5. Example. We will consider a uniform distribution of orientations ω_k, where the tensors $\langle\langle R\rangle\rangle_{km}$, $\langle\langle J_{km}\rangle\rangle_{km}$ are isotropic. Moreover, to simplify calculations we use a point approximation for the inclusions $S(|r|) = G(|r|)$ [4, 6], asymptotically exact as $|r| \to \infty$. Then for inclusions of one size, using the first terms of the series, we have

$$\langle\langle J_{12} - I - SR_2\rangle\rangle_{12} = \langle\langle SR_2SR_1\rangle\rangle_{12} = (3J_{12}^1, 2J_{12}^2),$$
$$\langle\langle T_{12} - SF_2\rangle\rangle_{12} = \langle\langle SR_2SF_1\rangle\rangle_{12} = (3T_{12}^1, 2T_{12}^2),$$

$$3J_{12}^1 = 2\xi^2(3\bar{k}_1)(2\bar{\mu}_2)|r|^{-6},$$
$$2J_{12}^2 = \frac{2}{5}\left[\xi^2(3\bar{k}_1)(2\bar{\mu}_2) + (2\bar{\mu}_1)(2\bar{\mu}_2)\left(7\gamma^2 - \frac{\eta^2}{4} + 2\xi\eta\right)\right]|r|^{-6},$$
$$\xi = (3k_0 + 4\mu_0)^{-1}, \quad \eta = (3\mu_0)^{-1}, \quad \gamma = -(3k_0 + 4\mu_0)(3\mu_0(3k_0 + 4\mu_0))^{-1},$$

where for the isotropic tensor $B_{ijk\ell}$

$$B = (3B^1, 2B^2) = 3B^1N_1 + 2B^2N_2; \quad N_1 = \frac{1}{3}\delta_{ij}\delta_{kl};$$
$$N_2 = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right);$$
$$\langle A_i\rangle L_0 \prod_{j=1}^{3} a_i^j \equiv (3\bar{k}_i, 2\bar{\mu}_i); \quad \langle A\rangle = \int A\psi(\omega)\,d\omega; \quad L_0 = (3k_0, 2\mu_0).$$

To obtain the expressions $3T_{12}^1$, $2T_{12}^2$ in the functions $3J_{12}^1$, $2J_{12}^2$ we replace $(3\bar{k}_i, 2\bar{\mu}_i)$ by $(3t_i^1, 2t_i^2) = \langle A_i\rangle q \prod_{j=1}^{3} a_i^j$.

For example, for spheroidal pores $(a^1 = a^2 = a \gg a^3)$ and $f(|r|) = n$

$$3t_1/p = \bar{k}/k_0 = \frac{4(1-\nu^2)}{3\pi(1-2\nu)}(a)^3, \quad \bar{\mu}/\mu_0 = \frac{8}{15\pi}\frac{(1-\nu)(5-\nu)}{(2-\nu)}(a)^3, \quad \nu = \frac{3k_0 - 2\mu_0}{2(3k_0 + \mu_0)}.$$

For spherical inclusions $(a^1 = a^2 = a^3 = a)$

$$3t_1/p = \bar{k}/k_0 = \frac{3k_0 + 4\mu_0}{4\mu_0}(a)^3, \quad \bar{\mu}/\mu_0 = \frac{5(3k_0 + 4\mu_0)}{9k_0 + 8\mu_0}(a)^3.$$

In the case of incompressible material $(\nu = 1/2,\ \beta^* = 1/L^*)$

$$3k^* = \frac{3\mu_0}{c_1}\left(1 - \frac{16}{15\pi^2}c_1\right), \quad 3k^* = \frac{4\mu_0}{c_2}\left(1 - \frac{29}{24}c_2\right), \quad c_i = \frac{4}{3}\pi(a)^3 n \qquad (5.1)$$

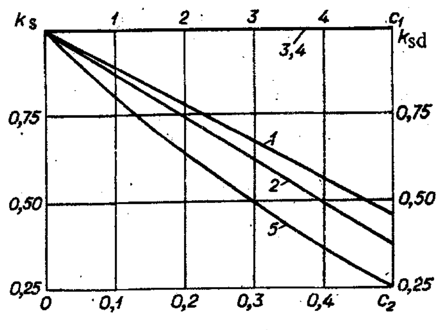

Fig. 1

for planar spheroidal and spherical pores; the values of c_i in the expressions presented have different physical meanings. Figure 1 shows normalized values of $k_{sd} = k^* c_1/\mu_0$, and $k_s = 3k^* c_2/4\mu_0$ for planar spheroidal and spherical inclusions, as calculated with Eq. (5.1) in curves 1, 2; curves 3, 4 are values of K_{sd} and K_s calculated with consideration of only two terms in expansion (3.3), as was done in [4]; curve 5 are values of K_s calculated by the method of [2].

We note that for an incompressible matrix ($\nu = 1/2$) and planar spheroidal pores, according to [2, 7] $k^* = k_0$ for any concentration c_1, which indicates the invalidity of the theory of [2, 7] in the case of limiting ν considered here.

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